Chapter 1 Complex Vectors

1.1 Complex algebra

A complex number is represented by $c = a + jb$, where a and b indicate the real and imaginary part of the complex number c. j is the imaginary number defined by $j^2 = -1$. The complex number can be written in phaser form as:

$$c = a + jb = |c|e^{j\phi} = |c|\cos\phi + j|c|\sin\phi \qquad (1.1.1)$$

where

$$|c| = \sqrt{a^2 + b^2} \qquad (1.1.2)$$

is the magnitude of c and

$$\phi = \tan^{-1}\left(\frac{b}{a}\right) \qquad (1.1.3)$$

is the phase of c.

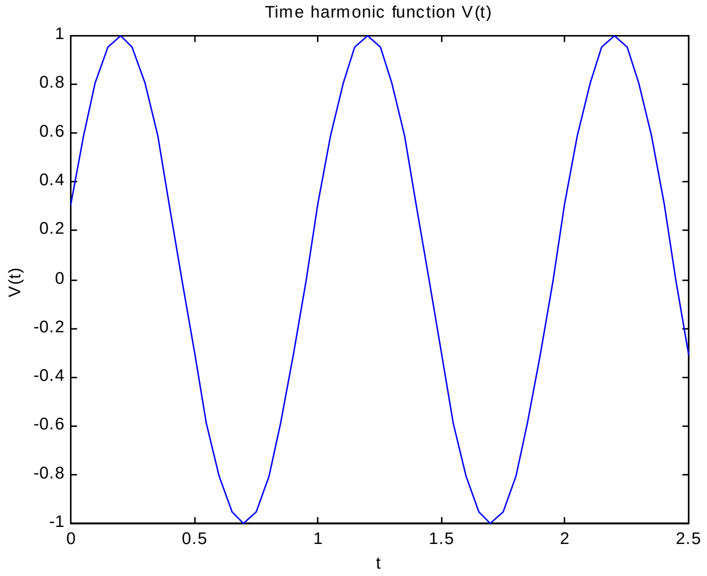
The complex conjugate of c is indicated by c^* and is defined by

$$c^* = a - jb \qquad (1.1.4)$$

1.2 Complex representation of Time-Harmonic scalars

Consider a time-harmonic real physical quantity $V(t)$ that varies sinusoidal with time.

$$V(t) = V_0 \cos(\omega t + \phi) \qquad (1.2.1)$$



where V_0 is the amplitude, ω is the angular frequency and ϕ is the phase of $V(t)$. Note that $\omega = 2\pi f$, where f is the frequency.

Fig.1.1.1 A time-harmonic function $V(t)$

The relation between the real physical quantity $V(t)$ and the phaser notation is directly related by

$$V(t) = Re\left\{Ve^{j\omega t}\right\} \qquad (1.2.2)$$

where $Re\{\ \}$ means taking the real part of the quantity in the braces $\{\ \}$ and $V = V_0 e^{j\phi}$. For simplicity, $Re\left\{(\ )e^{j\omega t}\right\}$ can be omitted in writing and thus,

$$V(t) \quad \text{(is equivalent to)} \quad \leftrightarrow \quad V \quad \text{(Phaser notation)} \qquad (1.2.3)$$

It should be noted that, $V(t)$ is a real function of t, but V is a complex function.

Using the above rule of equivalence, we find

$$V(t) + U(t) \leftrightarrow V + U \qquad (1.2.3)$$

It can also be shown that

$$\frac{\partial}{\partial t} V(t) \leftrightarrow j\omega V \qquad (1.2.4)$$

because

$$\frac{\partial}{\partial t} V(t) = -\omega V_0 \, sin(\omega t + \phi) = Re\left\{j\omega V_0 e^{j\phi} e^{j\omega t}\right\} \qquad (1.2.5)$$

Therefore, $j\omega$ can replace the tome derivative $\partial / \partial t$ in the complex representation of time-harmonic quantities.

The rule of equivalence of time-harmonic quantities should not be used carried out too far. For example,

$$V(t)U(t) \neq \text{(is not equivalent to)} VU \qquad (1.2.6)$$

1.3 **Time Average**

The time average value of a time-harmonic quantity is always zero.

$$\langle V(t) \rangle \equiv \frac{1}{T} \int_0^T V_0 \, cos(\omega t + \phi) dt = 0 \qquad (1.3.1)$$

However, the time-average value for the product of two-harmonic quantities is not always zero. For instance,

$$\langle V^2(t) \rangle \equiv \frac{1}{T} \int_0^T V_0^{\,2} \, cos^2(\omega t + \phi) dt = \frac{1}{T} \int_0^T V_0^{\,2} \left[\frac{1}{2} + \frac{1}{2} cos(2\omega t + 2\phi) \right] dt = \frac{V_0^{\,2}}{2} \qquad (1.3.4)$$

which is not zero.

Consider complex vectors $\boldsymbol{A} = \boldsymbol{A}_R + j\boldsymbol{A}_I$ and $\boldsymbol{B} = \boldsymbol{B}_R + j\boldsymbol{B}_I$. Their time-domain counterparts can be written as

$$\boldsymbol{A}(t) = \mathrm{Re}\left\{\boldsymbol{A} e^{j\omega t}\right\} = \boldsymbol{A}_R \cos \omega t - \boldsymbol{A}_I \sin \omega t \qquad (1.3.5)$$

$$\boldsymbol{B}(t) = \mathrm{Re}\left\{\boldsymbol{B}e^{j\omega t}\right\} = \boldsymbol{B}_R \cos\omega t - \boldsymbol{B}_I \sin\omega t \qquad (1.3.6)$$

The cross product of $\boldsymbol{A}(t)$ and $\boldsymbol{B}(t)$ then becomes

$$\boldsymbol{A}(t)\times\boldsymbol{B}(t) = \boldsymbol{A}_R\times\boldsymbol{B}_R\, cos^2(\omega t) + \boldsymbol{A}_I\times\boldsymbol{B}_I\, sin^2(\omega t) - \frac{1}{2}\left(\boldsymbol{A}_R\times\boldsymbol{B}_I + \boldsymbol{A}_I\times\boldsymbol{B}_R\right) sin\, 2\omega t$$

(1.3.7)

The time-average value of $\boldsymbol{A}(t)\times\boldsymbol{B}(t)$ is then

$$\langle\boldsymbol{A}(t)\times\boldsymbol{B}(t)\rangle = \frac{1}{2}\left(\boldsymbol{A}_R\times\boldsymbol{B}_R + \boldsymbol{A}_I\times\boldsymbol{B}_I\right) \qquad (1.3.8)$$

If we take the cross product of $\boldsymbol{A}$ and $\boldsymbol{B}^*$, we then get

$$\boldsymbol{A}\times\boldsymbol{B}^* = \boldsymbol{A}_R\times\boldsymbol{B}_R + \boldsymbol{A}_I\times\boldsymbol{B}_I + j(\boldsymbol{A}_I\times\boldsymbol{B}_R - \boldsymbol{A}_R\times\boldsymbol{B}_I) \quad (1.3.9)$$

We thus see that

$$\langle\boldsymbol{A}(t)\times\boldsymbol{B}(t)\rangle = \frac{1}{2} Re\left(\boldsymbol{A}\times\boldsymbol{B}^*\right) \qquad (1.3.10)$$

This rule will be very important when we encounter the concept of Poynting's power-density vector.